ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT
Linear Algebra and its Applications 414 (2006) 361–372
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Eigenvalues of second-order difference equations with coupled boundary conditions ☆

Huaqing Sun, Yuming Shi *

School of Mathematics and System Sciences, Shandong University, Jinan, Shandong 250100, PR China

Received 27 June 2005; accepted 15 October 2005
Available online 1 December 2005
Submitted by H. Schneider

Abstract

This paper is concerned with coupled boundary value problems for self-adjoint second-order difference equations. Existence of eigenvalues is proved, numbers of eigenvalues are calculated, and relationships between the eigenvalues of a self-adjoint second-order difference equation with three different coupled boundary conditions are established. These results extend the relevant existing results of periodic and anti-periodic boundary value problems.

AMS classification: 39A99; 34B05

Keywords: Self-adjoint second-order difference equation; Coupled boundary condition; Eigenvalue

1. Introduction

Consider the self-adjoint second-order difference equation

$$-\nabla(p_n \Delta y_n) + q_n y_n = \lambda w_n y_n, \quad n \in [0, N-1] \tag{1.1}$$

with the coupled boundary condition

☆ This research was supported by the NNSF of China (grant 10471077) and Shandong Research Funds for Young Scientists (grant 03BS094).
* Corresponding author. Tel.: +86 531 8363866; fax: +86 531 8564652.
E-mail addresses: ymshi@sdu.edu.cn (Y. Shi), sunhuaqing_2@163.com (H. Sun).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2005.10.012

$$\begin{pmatrix} y_{N-1} \\ \Delta y_{N-1} \end{pmatrix} = \mathrm{e}^{\mathrm{i}\alpha} K \begin{pmatrix} y_{-1} \\ \Delta y_{-1} \end{pmatrix}, \tag{1.2}$$

where $N \geqslant 2$ is an integer, Δ is the forward difference operator: $\Delta y_n = y_{n+1} - y_n$, and ∇ is the backward difference operator: $\nabla y_n = y_n - y_{n-1}$; p_n, q_n, and w_n are real numbers with $p_n > 0$ for $n \in [-1, N-1]$, $w_n > 0$ for $n \in [0, N-1]$, and $p_{-1} = p_{N-1} = 1$; λ is the spectral parameter; the interval $[0, N-1]$ is the integral set $\{n\}_{n=0}^{N-1}$; α, $-\pi < \alpha \leqslant \pi$, is a constant parameter; $\mathrm{i} = \sqrt{-1}$,

$$K = \begin{pmatrix} k_1 & 0 \\ k_2 & k_3 \end{pmatrix}, \quad k_j \in \mathbf{R}, \quad j = 1, 2, 3, \quad \text{with } k_1 k_3 = 1.$$

The boundary condition (1.2) contains the two special cases: the periodic and antiperiodic boundary conditions. In fact, (1.2) is the periodic boundary condition in the case where $\alpha = 0$ and $K = I$, the identity matrix, and (1.2) is the antiperiodic condition in the case where $\alpha = \pi$ and $K = I$. Eq. (1.1) with (1.2) is called a coupled boundary value problem.

We first briefly recall some relative existing results of eigenvalue problems for difference equations. Atkinson [2, Chapter 6, Section 2] discussed the boundary conditions

$$y_{-1} = \alpha y_{m-1}, \quad y_m = \beta y_0 \tag{1.3}$$

when he investigated the recurrence formula

$$c_n y_{n+1} = (a_n \hat{\lambda} + b_n) y_n - c_{n-1} y_{n-1}, \quad n \in [0, m-1], \tag{1.4}$$

where a_n, b_n, c_n, α, and β are real numbers, subject to $a_n > 0$, $c_n > 0$, and

$$\alpha c_{-1} = \beta c_{m-1}. \tag{1.5}$$

He remarked that all the eigenvalues of the boundary value problem (1.3) and (1.4) are real and they may not be all distinct. If $c_{-1} = c_{m-1}$ and $\alpha = \beta = 1$, he viewed the boundary conditions (1.3) as the periodic boundary conditions for (1.4). Shi and Chen [10] investigated the more general boundary value problem

$$-\nabla(C_n \Delta x_n) + B_n x_n = \lambda w_n x_n, \quad n \in [1, N], \quad N \geqslant 2, \tag{1.6}$$

$$R \begin{pmatrix} -x_0 \\ x_N \end{pmatrix} + S \begin{pmatrix} C_0 \Delta x_0 \\ C_N \Delta x_N \end{pmatrix} = 0, \tag{1.7}$$

where C_n, B_n, and w_n are $d \times d$ Hermitian matrices; C_0 and C_N are nonsingular; $w_n > 0$ for $n \in [1, N]$; and R and S are $2d \times 2d$ matrices. Moreover, R and S satisfy $\operatorname{rank}(R, S) = 2d$ and the self-adjoint condition $RS^* = SR^*$ [10, Lemma 2.1]. A series of spectral results were obtained. We shall remark that the boundary condition (1.7) includes the coupled boundary condition (1.2) when $d = 1$, and the boundary conditions (1.3) when (1.5) holds. More details will be discussed in the next section. Agarwal and Wong studied existence of minimal and maximal quasi-solutions of a second-order nonlinear periodic boundary value problem [1, Section 4]. Bohner [3] discussed disconjugacy for discrete linear Hamiltonian systems. In addition, spectral theory of discrete Hamiltonian systems on finite intervals was studied by Bohner [4] and Shi [11]. Recently, Bohner et al. [6] gave a relationship between the number of eigenvalues and the number of generalized zeros of principal solutions for symplectic difference systems with general boundary conditions. More recently, Wang and Shi [12] considered Eq. (1.1) with the periodic and antiperiodic boundary conditions. They found out the following very beautiful results (see [12, Theorems 2.2 and 3.1]): the periodic and antiperiodic boundary value problems have exactly N real eigenvalues $\{\lambda_i\}_{i=0}^{N-1}$ and $\{\tilde{\lambda}_i\}_{i=1}^{N}$, respectively, which satisfy

$$\lambda_0 < \tilde\lambda_1 \leqslant \tilde\lambda_2 < \lambda_1 \leqslant \lambda_2 < \tilde\lambda_3 \leqslant \tilde\lambda_4 < \cdots < \lambda_{N-2} \leqslant \lambda_{N-1} < \tilde\lambda_N, \quad \text{if } N \text{ is odd;}$$
$$\lambda_0 < \tilde\lambda_1 \leqslant \tilde\lambda_2 < \lambda_1 \leqslant \lambda_2 < \tilde\lambda_3 \leqslant \tilde\lambda_4 < \cdots < \tilde\lambda_{N-1} \leqslant \tilde\lambda_N < \lambda_{N-1}, \quad \text{if } N \text{ is even.}$$

These results are similar to those about eigenvalues of periodic and antiperiodic boundary value problems for second-order ordinary differential equations (cf. [7–9,13]).

Motivated by [12], we compare the eigenvalues of the eigenvalue problem (1.1) with the coupled boundary condition (1.2) as α varies and obtain relationships between the eigenvalues in the present paper. These results extend the above results obtained in [12]. We remark that Coddington and Levinson mainly used the Prüfer transformation of the second-order differential equations in their proof. Though the discrete Prüfer transformation has been established in [5], a similar method is difficultly employed in studying the discrete problem. So Wang and Shi used some oscillation results obtained by Atkinson [2] and some results obtained by Shi and Chen [10]. Similarly, in this paper, we will apply some results obtained by Shi and Chen [10] to prove the existence of eigenvalues of (1.1) and (1.2) and to calculate the number of these eigenvalues, and apply some oscillation results obtained by Atkinson [2] to compare the eigenvalues as α varies.

This paper is organized as follows. Section 2 gives some preliminaries including existence and numbers of eigenvalues of the coupled boundary value problems, a representation of solutions of a nonhomogeneous linear equation with initial conditions, and some properties of eigenvalues of a Dirichlet boundary value problem, which will be used in the next section. Section 3 pays attention to comparison between the eigenvalues of problem (1.1) and (1.2) as α varies.

2. Preliminaries

Eq. (1.1) can be rewritten as the recurrence formula

$$p_n y_{n+1} = (p_n + p_{n-1} + q_n - \lambda w_n) y_n - p_{n-1} y_{n-1}, \quad n \in [0, N-1]. \tag{2.1}$$

Clearly, y_n is a polynomial in λ with real coefficients since p_n, q_n, and w_n are all real. Hence, all the solutions of (1.1) are entire functions of λ. Especially, if $y_0 \neq 0$, y_n is a polynomial of degree n in λ for $n \leqslant N$. However, if $y_{-1} \neq 0$ and $y_0 = 0$, y_n is a polynomial of degree $n-1$ in λ for $n \leqslant N$.

We now prepare some results that are useful in the next section. The following lemma is Theorem 2.1 in [12]. It is also contained in (ii) of Remark 1 in [3].

Lemma 2.1 [12, Theorem 2.1]. *Let y and z be any solutions of* (1.1). *Then the Wronskian*

$$W[y,z](n) = \begin{vmatrix} y_{n+1} & z_{n+1} \\ p_n \Delta y_n & p_n \Delta z_n \end{vmatrix} = -p_n (y_{n+1} z_n - y_n z_{n+1}) \tag{2.2}$$

is a constant on $[-1, N-1]$.

Theorem 2.1. *The coupled boundary value problem* (1.1) *and* (1.2) *has exactly N real eigenvalues.*

Proof. By setting $d = 1$, $C_n = p_n$, $B_n = q_n$,

$$R = (R_1, R_2) = \begin{pmatrix} e^{i\alpha} k_1 & 1 \\ e^{i\alpha} k_2 & 0 \end{pmatrix}, \quad S = (S_1, S_2) = \begin{pmatrix} 0 & 0 \\ -e^{i\alpha} k_3 & 1 \end{pmatrix}, \tag{2.3}$$

shifting the whole interval $[1, N]$ left by one unit, and using $p_{-1} = p_{N-1} = 1$, (1.1) and (1.2) are written as (1.6) and (1.7), respectively. It is evident that $\operatorname{rank}(R, S) = 2d$ and $RS^* = SR^*$.

Hence, the boundary condition (1.2) is self-adjoint by [10, Lemma 2.1]. In addition, it follows from (2.3) and $C_{-1} = 1$ that

$$(R_1 + S_1 C_{-1}, S_2) = \begin{pmatrix} e^{i\alpha} k_1 & 0 \\ e^{i\alpha}(k_2 - k_3) & 1 \end{pmatrix}.$$

By noting that $k_1 \neq 0$, we get that $\operatorname{rank}(R_1 + S_1 C_{-1}, S_2) = 2$. Therefore, by [10, Theorem 4.1], the problem (1.1) and (1.2) has exactly N real eigenvalues. This completes the proof. □

Let $y_n(\lambda)$ be the solution of (1.1) with the initial conditions

$$y_{-1}(\lambda) = 0, \quad y_0(\lambda) \neq 0. \tag{2.4}$$

Consider the sequence

$$y_0(\lambda), y_1(\lambda), \ldots, y_{N-1}(\lambda). \tag{2.5}$$

If $y_n(\lambda) = 0$ for some $n \in (0, N-1)$, then, we get from (2.1) that $y_{n-1}(\lambda)$ and $y_{n+1}(\lambda)$ have opposite signs. Hence, we say that sequence (2.5) exhibits a change of sign if $y_n(\lambda) y_{n+1}(\lambda) < 0$ for some $n \in [0, N-1)$, or $y_n(\lambda) = 0$ for some $n \in (0, N-1)$.

Atkinson [2, Chapter 4] studied the boundary value problem (1.4) with the separated boundary conditions

$$y_{-1} = 0, \quad y_m + h y_{m-1} = 0, \tag{2.6}$$

where h is some fixed real number. Here, we consider (1.1) with the Dirichlet boundary conditions

$$y_{-1} = y_{N-1} = 0, \tag{2.7}$$

which will be used to compare the eigenvalues of (1.1) and (1.2) as α varies in the next section. By setting $p_n = c_n$, $p_n + p_{n-1} + q_n = b_n$, $w_n = a_n$ for $n \in [0, N-1]$, and $-\lambda = \hat{\lambda}$, (2.1) can be written as (1.4). So the following result for (1.1) and (2.7) can be directly derived from [2, Theorems 4.3.1 and 4.3.5] by setting $h = 0$ and $m = N - 1$.

Lemma 2.2 [12, Lemma 2.2]. *The boundary value problem* (1.1) *and* (2.7) *has exactly $N - 1$ real and simple eigenvalues, which can be arranged in the increasing order*

$$\mu_0 < \mu_1 < \cdots < \mu_{N-2}. \tag{2.8}$$

Let $y_n(\lambda)$ be the solution of (1.1) *with the initial conditions* (2.4). *Then sequence* (2.5) *exhibits no changes of sign for $\lambda < \mu_0$, exactly $r + 1$ changes of sign for $\mu_r < \lambda < \mu_{r+1}$ $(0 \leqslant r \leqslant N-3)$, and exactly $N - 1$ changes of sign for $\lambda > \mu_{N-2}$.*

Remark 2.1. In Lemma 2.2, the result that all the eigenvalues of (1.1) and (2.7) are real is also contained in (iii) of Remark 2 in [6].

Let φ_n and ψ_n be the solutions of (1.1) satisfying the following initial conditions:

$$\varphi_{-1} = \psi_0 = 1, \quad \varphi_0 = \psi_{-1} = 0, \tag{2.9}$$

respectively. By Lemma 2.1 and using $p_{N-1} = 1$, we have

$$\varphi_N \psi_{N-1} - \varphi_{N-1} \psi_N = -1. \tag{2.10}$$

Obviously, $\varphi_n(\lambda)$ and $\psi_n(\lambda)$ are two linearly independent solutions of (1.1). The following lemma can be derived from the proof of [12, Proposition 3.1].

Lemma 2.3. *Let μ_k $(0 \leqslant k \leqslant N-2)$ be the eigenvalues of (1.1) and (2.7) and be arranged as (2.8). Then, $\psi_n(\mu_k)$ is an eigenfunction of the problem (1.1) and (2.7) with respect to μ_k $(0 \leqslant k \leqslant N-2)$, that is, for $0 \leqslant k \leqslant N-2$, $\psi_n(\mu_k)$ is a nontrivial solution of (1.1) satisfying*

$$\psi_{-1}(\mu_k) = \psi_{N-1}(\mu_k) = 0. \tag{2.11}$$

Moreover, if k is odd, $\psi_N(\mu_k) > 0$ and if k is even, $\psi_N(\mu_k) < 0$ for $2 \leqslant k \leqslant N-2$.

A representation of solutions for a nonhomogeneous linear equation with initial conditions is given by the following lemma.

Lemma 2.4 [12, Theorem 2.3]. *For any $\{f_n\}_{n=0}^{N-1} \subset \mathbf{C}$ and for any $c_{-1}, c_0 \in \mathbf{C}$, the initial value problem*

$$-\nabla(p_n \Delta z_n) + (q_n - \lambda w_n) z_n = w_n f_n, \quad n \in [0, N-1],$$

$$z_{-1} = c_{-1}, \quad z_0 = c_0$$

has a unique solution z, which can be expressed as

$$z_n = c_{-1}\varphi_n + c_0\psi_n + \sum_{j=0}^{n-1} w_j(\varphi_n\psi_j - \varphi_j\psi_n) f_j, \quad n \in [-1, N],$$

where $\sum_{j=0}^{-2} \cdot = \sum_{j=0}^{-1} \cdot := 0$.

3. Main results

Let φ_n and ψ_n be defined in Section 2, and let $\lambda_j(e^{i\alpha}K)$ $(0 \leqslant j \leqslant N-1)$ be the eigenvalues of the coupled boundary value problem (1.1) and (1.2) and arranged in the nondecreasing order

$$\lambda_0(e^{i\alpha}K) \leqslant \lambda_1(e^{i\alpha}K) \leqslant \cdots \leqslant \lambda_{N-1}(e^{i\alpha}K).$$

Clearly, $\lambda_j(K)$ $(0 \leqslant j \leqslant N-1)$ denotes the eigenvalue of the problem (1.1) and (1.2) with $\alpha = 0$, and $\lambda_j(-K)$ $(0 \leqslant j \leqslant N-1)$ denotes the eigenvalue of the problem (1.1) and (1.2) with $\alpha = \pi$. We now present the main results of this paper.

Theorem 3.1. *Assume that $k_3 > 0$. Then, for every fixed $\alpha \neq 0$, $-\pi < \alpha < \pi$, we have the following inequalities:*

$$\begin{aligned}
\lambda_0(K) &< \lambda_0(e^{i\alpha}K) < \lambda_0(-K)\\
&\leqslant \lambda_1(-K) < \lambda_1(e^{i\alpha}K) < \lambda_1(K)\\
&\leqslant \lambda_2(K) < \lambda_2(e^{i\alpha}K) < \lambda_2(-K)\\
&\leqslant \lambda_3(-K) < \lambda_3(e^{i\alpha}K) < \lambda_3(K)\\
&\leqslant \cdots\\
&\leqslant \lambda_{N-2}(-K) < \lambda_{N-2}(e^{i\alpha}K) < \lambda_{N-2}(K)\\
&\leqslant \lambda_{N-1}(K) < \lambda_{N-1}(e^{i\alpha}K) < \lambda_{N-1}(-K) \quad \textit{if } N \textit{ is odd},
\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
&\lambda_0(K) < \lambda_0(e^{i\alpha}K) < \lambda_0(-K)\\
&\quad \leqslant \lambda_1(-K) < \lambda_1(e^{i\alpha}K) < \lambda_1(K)\\
&\quad \leqslant \lambda_2(K) < \lambda_2(e^{i\alpha}K) < \lambda_2(-K)\\
&\quad \leqslant \lambda_3(-K) < \lambda_3(e^{i\alpha}K) < \lambda_3(K)\\
&\quad \leqslant \ldots\\
&\quad \leqslant \lambda_{N-2}(K) < \lambda_{N-2}(e^{i\alpha}K) < \lambda_{N-2}(-K)\\
&\quad \leqslant \lambda_{N-1}(-K) < \lambda_{N-1}(e^{i\alpha}K) < \lambda_{N-1}(K) \quad \text{if } N \text{ is even.}
\end{aligned}
\tag{3.2}
$$

For every fixed $\alpha \neq 0, -\pi < \alpha < \pi$, $\lambda_j(e^{i\alpha}K)$ $(0 \leqslant j \leqslant N-1)$ is a simple eigenvalue of (1.1) *and* (1.2). *Whether N is odd or even, $\lambda_0(K)$ is a simple eigenvalues of* (1.1) *and* (1.2) *with $\alpha = 0$. For some odd number j $(1 \leqslant j \leqslant N-2)$, if $\lambda_j(K) < \lambda_{j+1}(K)$, then $\lambda_j(K)$ and $\lambda_{j+1}(K)$ are simple eigenvalues of* (1.1) *and* (1.2) *with $\alpha = 0$; however, if $\lambda_j(K) = \lambda_{j+1}(K)$, then $\lambda_j(K)$ is a multiple eigenvalue of* (1.1) *and* (1.2) *with $\alpha = 0$. Similar results hold for the cases $\lambda_j(-K) < \lambda_{j+1}(-K)$ and $\lambda_j(-K) = \lambda_{j+1}(-K)$ for some even number j $(0 \leqslant j \leqslant N-2)$. Especially, if N is odd, $\lambda_{N-1}(-K)$ is a simple eigenvalue of* (1.1) *and* (1.2) *with $\alpha = \pi$ and if N is even, $\lambda_{N-1}(K)$ is a simple eigenvalue of* (1.1) *and* (1.2) *with $\alpha = 0$.*

Remark 3.1. If $k_3 < 0$, a similar results can be obtained by applying Theorem 3.1 to $-K$. In fact, $e^{i\alpha}K = e^{i(\pi+\alpha)}(-K)$ for $\alpha \in (-\pi, 0)$ and $e^{i\alpha}K = e^{i(-\pi+\alpha)}(-K)$ for $\alpha \in (0, \pi)$. Hence, the boundary condition (1.2) in the case of $k_3 < 0$ and $\alpha \neq 0, -\pi < \alpha < \pi$, can be written as condition (1.2), where α is replaced by $\pi + \alpha$ for $\alpha \in (-\pi, 0)$ and $-\pi + \alpha$ for $\alpha \in (0, \pi)$, and K is replaced by $-K$.

Remark 3.2. Theorem 3.1 extends [12, Theorem 3.1].

Before proving Theorem 3.1, we prove the following seven propositions.

Proposition 3.1. *For $\lambda \in \mathbf{C}$, λ is an eigenvalue of* (1.1) *and* (1.2) *if and only if*

$$f(\lambda) = 2\cos\alpha, \tag{3.3}$$

where

$$f(\lambda) := k_3\varphi_{N-1}(\lambda) + k_1\,\Delta\psi_{N-1}(\lambda) - (k_2 - k_3)\psi_{N-1}(\lambda).$$

Moreover, λ is a multiple eigenvalue of (1.1) *and* (1.2) *if and only if*

$$
\begin{aligned}
&\varphi_{N-1}(\lambda) = e^{i\alpha}k_1, \quad \Delta\varphi_{N-1}(\lambda) = e^{i\alpha}(k_2 - k_3),\\
&\psi_{N-1}(\lambda) = 0, \quad \Delta\psi_{N-1}(\lambda) = e^{i\alpha}k_3.
\end{aligned}
\tag{3.4}
$$

Proof. Since φ_n and ψ_n are linearly independent solutions of (1.1), then λ is an eigenvalue of the problem (1.1) and (1.2) if and only if there exist two constants C_1 and C_2 not both zero such that $C_1\varphi_n + C_2\psi_n$ satisfies (1.2), which yields

$$\begin{pmatrix} \varphi_{N-1}(\lambda) - e^{i\alpha}k_1 & \psi_{N-1}(\lambda) \\ \Delta\varphi_{N-1}(\lambda) - e^{i\alpha}(k_2 - k_3) & \Delta\psi_{N-1}(\lambda) - e^{i\alpha}k_3 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0. \tag{3.5}$$

It is evident that (3.5) has a nontrivial solution (C_1, C_2) if and only if

$$\det\begin{pmatrix} \varphi_{N-1}(\lambda) - \mathrm{e}^{\mathrm{i}\alpha}k_1 & \psi_{N-1}(\lambda) \\ \Delta\varphi_{N-1}(\lambda) - \mathrm{e}^{\mathrm{i}\alpha}(k_2 - k_3) & \Delta\psi_{N-1}(\lambda) - \mathrm{e}^{\mathrm{i}\alpha}k_3 \end{pmatrix} = 0,$$

which, together with (2.10) and $k_1k_3 = 1$, implies that

$$1 + \mathrm{e}^{2\mathrm{i}\alpha} - \mathrm{e}^{\mathrm{i}\alpha} f(\lambda) = 0.$$

Then (3.3) follows from the above relation and the fact that $\mathrm{e}^{-\mathrm{i}\alpha} + \mathrm{e}^{\mathrm{i}\alpha} = 2\cos\alpha$. On the other hand, (1.1) has two linearly independent solutions satisfying (1.2) if and only if all the entries of the coefficient matrix of (3.5) are zero. Hence, λ is a multiple eigenvalue of (1.1) and (1.2) if and only if (3.4) holds. This completes the proof. □

The following result is a direct consequence of the first result of Proposition 3.1.

Corollary 3.1. *For any $\alpha \in (-\pi, \pi]$,*

$$\lambda_j(\mathrm{e}^{\mathrm{i}\alpha}K) = \lambda_j(\mathrm{e}^{-\mathrm{i}\alpha}K), \quad 0 \leqslant j \leqslant N-1.$$

Proposition 3.2. *Assume that $k_3 > 0$. For each k, $0 \leqslant k \leqslant N-2$, $f(\mu_k) \geqslant 2$ if k is odd, and $f(\mu_k) \leqslant -2$ if k is even.*

Proof. It follows from (2.10) and (2.11) that for each k, $0 \leqslant k \leqslant N-2$,

$$\varphi_{N-1}(\mu_k)\psi_N(\mu_k) = 1. \tag{3.6}$$

From (2.11) and (3.6), and by the definition of $f(\lambda)$, we get

$$f(\mu_k) = \frac{k_3}{\psi_N(\mu_k)} + k_1\psi_N(\mu_k).$$

Hence, noting $k_1k_3 = 1$ and $k_3 > 0$, and by Lemma 2.3, we have that if k is odd, then

$$f(\mu_k) \geqslant 2$$

and if k is even, then

$$f(\mu_k) \leqslant -2.$$

This completes the proof. □

Since φ_n and ψ_n are both polynomials in λ, so is $f(\lambda)$. Denote

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}f(\lambda) := f'(\lambda), \qquad \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}f(\lambda) := f''(\lambda).$$

Proposition 3.3. *Assume that $k_3 > 0$. Equations $f'(\lambda) = 0$ and $f(\lambda) = 2$ or -2 hold if and only if λ is a multiple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ or $\alpha = \pi$. If $f(\lambda) = 2$ or -2 for some $\lambda = \mu_i$ $(0 \leqslant i \leqslant N-2)$, then λ is a multiple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ or $\alpha = \pi$. If $f(\lambda) = 2$ or -2 for some $\lambda \neq \mu_i$ $(0 \leqslant i \leqslant N-2)$, then λ is a simple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ or $\alpha = \pi$ and for such a λ,*

$$\begin{aligned} &f'(\lambda) < 0, \quad \lambda < \mu_0; \\ &(-1)^r f'(\lambda) > 0, \quad \mu_r < \lambda < \mu_{r+1}, \quad 0 \leqslant r \leqslant N-3; \\ &(-1)^{N-2} f'(\lambda) > 0, \quad \lambda > \mu_{N-2}. \end{aligned}$$

Proof. Since φ_n and ψ_n are solutions of (1.1), we have

$$-\nabla\Big(p_n\,\Delta\varphi_n(\lambda)\Big)+q_n\varphi_n(\lambda)=\lambda w_n\varphi_n(\lambda), \tag{3.7}$$

$$-\nabla\Big(p_n\,\Delta\psi_n(\lambda)\Big)+q_n\psi_n(\lambda)=\lambda w_n\psi_n(\lambda). \tag{3.8}$$

Differentiating (3.7) and (3.8) with respect to λ, respectively, yields that

$$-\nabla\Big(p_n\,\Delta\varphi_n'(\lambda)\Big)+(q_n-\lambda w_n)\varphi_n'(\lambda)=w_n\varphi_n(\lambda), \tag{3.9}$$

$$-\nabla\Big(p_n\,\Delta\psi_n'(\lambda)\Big)+(q_n-\lambda w_n)\psi_n'(\lambda)=w_n\psi_n(\lambda). \tag{3.10}$$

It follows from (2.9) that

$$\varphi_0'=\varphi_{-1}'=\psi_0'=\psi_{-1}'=0. \tag{3.11}$$

Thus, by Lemma 2.4 and from (3.9)–(3.11), we have

$$\begin{aligned}\varphi_n'(\lambda)&=\sum_{j=0}^{n-1}w_j\varphi_j(\lambda)\Big(\varphi_n(\lambda)\psi_j(\lambda)-\varphi_j(\lambda)\psi_n(\lambda)\Big),\\ \psi_n'(\lambda)&=\sum_{j=0}^{n-1}w_j\psi_j(\lambda)\Big(\varphi_n(\lambda)\psi_j(\lambda)-\varphi_j(\lambda)\psi_n(\lambda)\Big).\end{aligned} \tag{3.12}$$

The second relation in (3.12) implies that

$$\Delta\psi_{n-1}'(\lambda)=\sum_{j=0}^{n-1}w_j\psi_j(\lambda)\Big(\Delta\varphi_{n-1}(\lambda)\psi_j(\lambda)-\varphi_j(\lambda)\Delta\psi_{n-1}(\lambda)\Big).$$

Hence, not indicating λ explicitly, we get

$$\begin{aligned}f'&=k_3\varphi_{N-1}'+k_1\,\Delta\psi_{N-1}'-(k_2-k_3)\psi_{N-1}'\\ &=k_3\sum_{j=0}^{N-2}w_j\varphi_j\Big(\varphi_{N-1}\psi_j-\varphi_j\psi_{N-1}\Big)+k_1\sum_{j=0}^{N-1}w_j\psi_j\Big(\Delta\varphi_{N-1}\psi_j-\varphi_j\,\Delta\psi_{N-1}\Big)\\ &\quad-(k_2-k_3)\sum_{j=0}^{N-2}w_j\psi_j\Big(\varphi_{N-1}\psi_j-\varphi_j\psi_{N-1}\Big)\\ &=\sum_{j=0}^{N-1}w_j\delta_j,\end{aligned}$$

where

$$\begin{aligned}\delta_j:=&\Big(k_1\,\Delta\varphi_{N-1}-(k_2-k_3)\varphi_{N-1}\Big)\psi_j^2\\ &+\Big(k_3\varphi_{N-1}-k_1\,\Delta\psi_{N-1}+(k_2-k_3)\psi_{N-1}\Big)\psi_j\varphi_j-k_3\psi_{N-1}\varphi_j^2\\ =&\,(\psi_j,\varphi_j)I\begin{pmatrix}\psi_j\\ \varphi_j\end{pmatrix}\end{aligned}$$

and

$$I := \begin{pmatrix} k_1\,\Delta\varphi_{N-1} - (k_2-k_3)\varphi_{N-1} & \dfrac{k_3\varphi_{N-1} - k_1\,\Delta\psi_{N-1} + (k_2-k_3)\psi_{N-1}}{2} \\ \dfrac{k_3\varphi_{N-1} - k_1\,\Delta\psi_{N-1} + (k_2-k_3)\psi_{N-1}}{2} & -k_3\psi_{N-1} \end{pmatrix},$$

which is symmetric for any $\lambda \in \mathbf{R}$. Then, we have

$$\begin{aligned} \det I(\lambda) &= -k_3\psi_{N-1}(\lambda)\Big(k_1\,\Delta\varphi_{N-1}(\lambda) - (k_2-k_3)\varphi_{N-1}(\lambda)\Big) \\ &\quad - \frac{\Big(k_3\varphi_{N-1}(\lambda) - k_1\,\Delta\psi_{N-1}(\lambda) + (k_2-k_3)\psi_{N-1}(\lambda)\Big)^2}{4} \\ &= -\frac{1}{4}f^2(\lambda) + 1. \end{aligned} \tag{3.13}$$

Hence, if $f(\lambda) = 2$ or -2, we get from (3.13) that $\det I(\lambda) = 0$. Then, for any fixed λ with $f(\lambda) = 2$ or -2, the matrix $I(\lambda)$ is positive semi-definite or negative semi-definite. Therefore, for such a λ, $f'(\lambda)$ cannot vanish unless $\delta_j(\lambda) = 0$ for all $0 \leqslant j \leqslant N-1$. Because φ_n and ψ_n are linearly independent, $\delta_j(\lambda)$ is identically zero if and only if all the entries of the matrix $I(\lambda)$ vanish, namely,

$$\begin{cases} k_3\psi_{N-1}(\lambda) = 0, \\ k_1\,\Delta\varphi_{N-1}(\lambda) - (k_2-k_3)\varphi_{N-1}(\lambda) = 0, \\ k_3\varphi_{N-1}(\lambda) - k_1\,\Delta\psi_{N-1}(\lambda) + (k_2-k_3)\psi_{N-1}(\lambda) = 0, \end{cases} \tag{3.14}$$

which, together with $f(\lambda) = 2$ and $k_1k_3 = 1$, implies

$$\varphi_{N-1}(\lambda) = k_1, \quad \Delta\varphi_{N-1}(\lambda) = k_2 - k_3, \quad \psi_{N-1}(\lambda) = 0, \quad \Delta\psi_{N-1}(\lambda) = k_3. \tag{3.15}$$

Then by Proposition 3.1, λ is a multiple eigenvalue of (1.1) and (1.2) with $\alpha = 0$. In addition, (3.14), together with $f(\lambda) = -2$ and $k_1k_3 = 1$, implies

$$\varphi_{N-1}(\lambda) = -k_1, \quad \Delta\varphi_{N-1}(\lambda) = -(k_2 - k_3), \quad \psi_{N-1}(\lambda) = 0, \quad \Delta\psi_{N-1}(\lambda) = -k_3. \tag{3.16}$$

Then by Proposition 3.1, λ is a multiple eigenvalue of (1.1) and (1.2) with $\alpha = \pi$. Conversely, from (3.15) or (3.16), it can be easily verified that (3.14) holds, then $f'(\lambda) = 0$. It follows again from (3.15) or (3.16) that $f(\lambda) = 2$ or $f(\lambda) = -2$. Thus $f'(\lambda) = 0$ and $f(\lambda) = 2$ or -2 if and only if λ is a multiple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ or $\alpha = \pi$.

Further, for every fixed λ with $f(\lambda) = 2$ or -2, (3.13) implies that

$$\begin{aligned} &-k_3\psi_{N-1}(\lambda)\Big(k_1\,\Delta\varphi_{N-1}(\lambda) - (k_2-k_3)\varphi_{N-1}(\lambda)\Big) \\ &\quad = \frac{\Big(k_3\varphi_{N-1}(\lambda) - k_1\,\Delta\psi_{N-1}(\lambda) + (k_2-k_3)\psi_{N-1}(\lambda)\Big)^2}{4}. \end{aligned} \tag{3.17}$$

Therefore, from (3.17) and by the definition of δ_j, we have

$$\delta_j = -k_3\psi_{N-1}(\lambda)\Big(\varphi_j(\lambda) - \frac{k_3\varphi_{N-1}(\lambda) - k_1\,\Delta\psi_{N-1}(\lambda) + (k_2-k_3)\psi_{N-1}(\lambda)}{2k_3\psi_{N-1}(\lambda)}\psi_j(\lambda)\Big)^2$$

and consequently, not indicating λ explicitly, we have

$$f' = -k_3\psi_{N-1}\sum_{j=0}^{N-1} w_j\Big(\varphi_j - \frac{k_3\varphi_{N-1} - k_1\,\Delta\psi_{N-1} + (k_2-k_3)\psi_{N-1}}{2k_3\psi_{N-1}}\psi_j\Big)^2 \tag{3.18}$$

for every fixed λ with $f(\lambda) = 2$ or -2.

Suppose that $f(\lambda) = 2$ or -2 for some $\lambda = \mu_j$ $(0 \leqslant j \leqslant N-2)$, we have $\psi_{N-1}(\lambda) = 0$. Then it follows from (3.18) that $f'(\lambda) = 0$. From the above discussions, λ is a multiple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ or $\alpha = \pi$.

Suppose that $f(\lambda) = 2$ or -2 for some $\lambda \neq \mu_j$ $(0 \leqslant j \leqslant N-2)$, we have $\psi_{N-1}(\lambda) \neq 0$. From the above discussions again, λ is a simple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ or $\alpha = \pi$, and δ_j is not identically zero for $0 \leqslant j \leqslant N-1$. For this λ, (3.18) and $k_3 > 0$ imply that $f'(\lambda)$ and $-\psi_{N-1}(\lambda)$ have the same sign. By Lemma 2.2 and from $\psi_0 = 1 > 0$, it follows that $-\psi_{N-1}(\lambda) < 0$ for $\lambda < \mu_0$; $\operatorname{sgn}(-\psi_{N-1}(\lambda)) = \operatorname{sgn}((-1)^{r+1}(-1)) = \operatorname{sgn}(-1)^r$ for $\mu_r < \lambda < \mu_{r+1}, 0 \leqslant r \leqslant N-3$; and $\operatorname{sgn}(-\psi_{N-1}(\lambda)) = \operatorname{sgn}(-1)^{N-2}$ for $\lambda > \mu_{N-2}$. The entire proof is complete. □

Proposition 3.4. *For any fixed $\alpha \neq 0, -\pi < \alpha < \pi$, each eigenvalue of* (1.1) *and* (1.2) *is simple.*

Proof. Fix α, $-\pi < \alpha < \pi$ with $\alpha \neq 0$. Suppose that λ is an eigenvalue of the problem (1.1) and (1.2). By Proposition 3.1, we have $f^2(\lambda) = 4\cos^2\alpha < 4$. It follows from (3.13) that $\det I(\lambda) > 0$ and the matrix $I(\lambda)$ is positive definite or negative definite. Hence, $\delta_j > 0$ for $0 \leqslant j \leqslant N-1$ or $\delta_j < 0$ for $0 \leqslant j \leqslant N-1$ since φ_n and ψ_n are linearly independent.

If λ is a multiple eigenvalue of problem (1.1) and (1.2), then (3.4) holds by Proposition 3.1. By using (3.4), it can be easily verified that (3.14) holds, that is, all the entries of the matrix $I(\lambda)$ are zero. Then $\delta_j = 0$ for $0 \leqslant j \leqslant N-1$, which is contrary to $\delta_j \neq 0$ for $0 \leqslant j \leqslant N-1$. Hence, λ is a simple eigenvalue of (1.1) and (1.2). This completes the proof. □

Proposition 3.5. *Assume that $k_3 > 0$. There exists a constant ν_0 such that $\nu_0 < \mu_0$ and $f(\nu_0) \geqslant 2$.*

Proof. By the discussions in the first paragraph in Section 2, $\varphi_{N-1}(\lambda)$ is a polynomial of degree $N-2$ in λ, $\psi_N(\lambda)$ is a polynomial of degree N in λ, and $\psi_{N-1}(\lambda)$ is a polynomial of degree $N-1$ in λ. Further, $\psi_N(\lambda)$ can be written as

$$\psi_N(\lambda) = (-1)^N A_N\lambda^N + A_{N-1}\lambda^{N-1} + \cdots + A_0,$$

where $A_N = w_0w_1\ldots w_{N-1}(p_0p_1\ldots p_{N-1})^{-1} > 0$ and A_n is a certain constant for $n \in [0, N-1]$. Then

$$f(\lambda) = (-1)^N k_1 A_N\lambda^N + h(\lambda), \tag{3.19}$$

where $h(\lambda)$ is a polynomial in λ whose degree is not larger than $N-1$. Clearly, as $\lambda \to -\infty$, $f(\lambda) \to +\infty$ since $k_3 > 0$ and $k_1k_3 = 1$. By Propositions 3.2, $f(\mu_0) \leqslant -2$. So there exists a constant $\nu_0 < \mu_0$ such that $f(\nu_0) \geqslant 2$. The proof is complete. □

Proposition 3.6. *Assume that $k_3 > 0$. If N is odd, there exists a constant ξ_0 such that $\mu_{N-2} < \xi_0$ and $f(\xi_0) \leqslant -2$, and if N is even, there exists a constant η_0 such that $\mu_{N-2} < \eta_0$ and $f(\eta_0) \geqslant 2$.*

Proof. By Proposition 3.2, if N is odd, $f(\mu_{N-2}) \geqslant 2$ and if N is even, $f(\mu_{N-2}) \leqslant -2$. It follows from (3.19) that if N is odd, $f(\lambda) \to -\infty$ as $\lambda \to +\infty$; and if N is even, $f(\lambda) \to +\infty$

as $\lambda \to +\infty$. Hence, if N is odd, there exists a constant $\xi_0 > \mu_{N-2}$ such that $f(\xi_0) \leqslant -2$; if N is even, there exists a constant $\eta_0 > \mu_{N-2}$ such that $f(\eta_0) \geqslant 2$. This completes the proof. $\square$

Proposition 3.7. *Assume that $k_3 > 0$. If k is odd, $f(\mu_k) = 2$, and $f'(\mu_k) = 0$, then $f''(\mu_k) < 0$; and if k is even, $f(\mu_k) = -2$, and $f'(\mu_k) = 0$, then $f''(\mu_k) > 0$ for $0 \leqslant k \leqslant N-2$.*

Proof. We first prove the first result. Suppose that k is odd, $f(\mu_k) = 2$, and $f'(\mu_k) = 0$. Then μ_k is a multiple eigenvalue of (1.1) and (1.2) with $\alpha = 0$ by Proposition 3.3. Then by Proposition 3.1, (3.4) holds for $\lambda = \mu_k$ and $\alpha = 0$, that is,

$$\varphi_{N-1}(\mu_k) = k_1, \quad \Delta\varphi_{N-1}(\mu_k) = k_2 - k_3, \quad \psi_{N-1}(\mu_k) = 0, \quad \Delta\psi_{N-1}(\mu_k) = k_3. \tag{3.20}$$

Differentiating $f(\lambda)$ with respect to λ two times, we get

$$f''(\mu_k) = k_3\varphi''_{N-1}(\mu_k) + k_1\,\Delta\psi''_{N-1}(\mu_k) - (k_2 - k_3)\psi''_{N-1}(\mu_k). \tag{3.21}$$

Differentiating (2.10) with respect to λ two times and from (3.20), we get

$$\begin{aligned}&-\Big(k_3\varphi''_{N-1}(\mu_k) + k_1\,\Delta\psi''_{N-1}(\mu_k) - (k_2 - k_3)\psi''_{N-1}(\mu_k)\Big)\\ &\quad+2\Big(\varphi'_N(\mu_k)\psi'_{N-1}(\mu_k) - \varphi'_{N-1}(\mu_k)\psi'_N(\mu_k)\Big) = 0,\end{aligned}$$

which, together with (3.21), implies that

$$f''(\mu_k) = 2\Big(\varphi'_N(\mu_k)\psi'_{N-1}(\mu_k) - \varphi'_{N-1}(\mu_k)\psi'_N(\mu_k)\Big). \tag{3.22}$$

On the other hand, it follows from (3.12) and (2.10) that, not indicating μ_k explicitly,

$$\begin{aligned}&\varphi'_N\psi'_{N-1} - \varphi'_{N-1}\psi'_N\\ &\quad= \sum_{j=0}^{N-1} w_j\varphi_j\Big(\varphi_N\psi_j - \varphi_j\psi_N\Big)\sum_{j=0}^{N-2} w_j\psi_j\Big(\varphi_{N-1}\psi_j - \varphi_j\psi_{N-1}\Big)\\ &\qquad- \sum_{j=0}^{N-2} w_j\varphi_j\Big(\varphi_{N-1}\psi_j - \varphi_j\psi_{N-1}\Big)\sum_{j=0}^{N-1} w_j\psi_j\Big(\varphi_N\psi_j - \varphi_j\psi_N\Big)\\ &\quad= \left(\sum_{j=0}^{N-1} w_j\varphi_j\psi_j\right)^2 - \sum_{j=0}^{N-1} w_j\varphi_j^2\sum_{j=0}^{N-1} w_j\psi_j^2.\end{aligned}$$

Since φ_n and ψ_n are linearly independent on $[-1, N]$, the above relation implies that $f''(\mu_k) < 0$ by Hölder's inequality, which proves the first conclusion.

The second conclusion can be shown similarly. Hence, the proof is complete. $\square$

Remark 3.3. Let $K = I$, that is, $k_1 = k_3 = 1$, $k_2 = 0$. Then $f(\lambda) = \varphi_{N-1}(\lambda) + \psi_N(\lambda)$. In this case, Propositions 3.2, 3.5–3.7 are the same as Propositions 3.1, 3.3–3.5 in [12], respectively, and most of the results of Proposition 3.3 are the same as the results of Proposition 3.2 in [12].

Finally, we turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. Fix $\alpha \neq 0, -\pi < \alpha < \pi$. By Propositions 3.1–3.7, Theorem 2.1, and the intermediate value theorem, we can conclude that if N is odd,

$$\begin{aligned}
\nu_0 &\leqslant \lambda_0(K) < \lambda_0(\mathrm{e}^{\mathrm{i}\alpha}K) < \lambda_0(-K)\\
&\leqslant \mu_0 \leqslant \lambda_1(-K) < \lambda_1(\mathrm{e}^{\mathrm{i}\alpha}K) < \lambda_1(K) \leqslant \mu_1\\
&\leqslant \lambda_2(K) < \lambda_2(\mathrm{e}^{\mathrm{i}\alpha}K) < \lambda_2(-K)\\
&\leqslant \mu_2 \leqslant \lambda_3(-K) < \lambda_3(\mathrm{e}^{\mathrm{i}\alpha}K) < \lambda_3(K) \leqslant \mu_3\\
&\leqslant \dots\\
&\leqslant \mu_{N-3} \leqslant \lambda_{N-2}(K) < \lambda_{N-2}(\mathrm{e}^{\mathrm{i}\alpha}K) < \lambda_{N-2}(-K)\\
&\leqslant \mu_{N-2} \leqslant \lambda_{N-1}(-K) < \lambda_{N-1}(\mathrm{e}^{\mathrm{i}\alpha}K) < \lambda_{N-1}(-K) \leqslant \xi_0,
\end{aligned}$$

and if N is even,

$$\begin{aligned}
\nu_0 &\leqslant \lambda_0(K) < \lambda_0(\mathrm{e}^{\mathrm{i}\alpha}K) < \lambda_0(-K)\\
&\leqslant \mu_0 \leqslant \lambda_1(-K) < \lambda_1(\mathrm{e}^{\mathrm{i}\alpha}K) < \lambda_1(K) \leqslant \mu_1\\
&\leqslant \lambda_2(K) < \lambda_2(\mathrm{e}^{\mathrm{i}\alpha}K) < \lambda_2(-K)\\
&\leqslant \mu_2 \leqslant \lambda_3(-K) < \lambda_3(\mathrm{e}^{\mathrm{i}\alpha}K) < \lambda_3(K) \leqslant \mu_3\\
&\leqslant \dots\\
&\leqslant \mu_{N-3} \leqslant \lambda_{N-2}(K) < \lambda_{N-2}(\mathrm{e}^{\mathrm{i}\alpha}K) < \lambda_{N-2}(-K)\\
&\leqslant \mu_{N-2} \leqslant \lambda_{N-1}(-K) < \lambda_{N-1}(\mathrm{e}^{\mathrm{i}\alpha}K) < \lambda_{N-1}(K) \leqslant \eta_0,
\end{aligned}$$

and consequently, Theorem 3.1 holds. This completes the proof. □

References

[1] R.P. Agarwal, P.J.Y. Wong, Advanced Topics in Difference Equations, Kluwer Academic, Dordrecht, 1997.
[2] F.V. Atkinson, Discrete and Continuous Boundary Problems, Academic Press, New York, 1964.
[3] M. Bohner, Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions, J. Math. Anal. Appl. 199 (3) (1996) 804–826.
[4] M. Bohner, Discrete linear Hamiltonian eigenvalue problems, Comput. Math. Appl. 36 (1998) 179–192.
[5] M. Bohner, O. Došlý, The discrete Prüfer transformation, Proc. Amer. Math. Soc. 129 (2001) 2715–2725.
[6] M. Bohner, O. Došlý, W. Kratz, An oscillation theorem for discrete eigenvalue problems, Rocky Mountain J. Math. 33 (2003) 1233–1260.
[7] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.
[8] J.K. Hale, Ordinary Differential Equations, Wiley, New York, 1969.
[9] W. Magnus, S. Winkler, Hill's Equations, Wiley, New York, 1966.
[10] Y. Shi, S. Chen, Spectral theory of second-order vector difference equations, J. Math. Anal. Appl. 239 (1999) 195–212.
[11] Y. Shi, Spectral theory of discrete linear Hamiltonian systems, J. Math. Anal. Appl. 289 (2004) 554–570.
[12] Y. Wang, Y. Shi, Eigenvalues of second-order difference equations with periodic and antiperiodic boundary conditions, J. Math. Anal. Appl. 309 (2005) 56–69.
[13] M. Zhang, The rotation number approach to eigenvalues of the one-dimensional p-Laplacian with periodic potentials, J. London Math. Soc. 64 (2001) 125–143.